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Fixed points of decreasing operators in ordered Banach spaces and applications to nonlinear second order elliptic equations[☆]

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ABSTRACT

In this paper, we consider some decreasing operators in ordered Banach spaces. We study the existence and uniqueness of fixed points and properties of the iterative sequences for these operators. Lastly, the results are applied to nonlinear second order elliptic equations.

1. Introduction

Among investigations of nonlinear operators in ordered Banach spaces, there are more results of increasing operators than those of decreasing operators. In this paper, we investigate decreasing condensing operators in ordered Banach spaces. We weaken the conditions of the known results, study the limits of some iterative sequences, and obtain the existence and uniqueness of fixed points for the operators. Lastly, we apply the results to the first boundary value problem for nonlinear second order elliptic equations.

Throughout this paper, E is a real Banach space with norm $\|\cdot\|$, θ is zero in E, and P is a cone in E. So, a partially ordered relation in E is given by $x \le y$ iff $y - x \in P$. We write

$$P^+ = P \setminus \{\theta\}, \qquad S_-(a) = \{x \in E | x \le a\}, \qquad S_+(a) = \{x \in E | x \ge a\}, \qquad [a, b] = S_+(a) \cap S_-(b).$$

A cone $P \subset E$ is said to be normal if there exists a constant N, such that $\theta \le x \le y \Rightarrow \|x\| \le N\|y\|$. Cone $P \subset E$ is said to be minihedral if $\sup\{x, y\}$ exists for any pair $\{x, y\}$.

Let $D \subset E$. An operator $A: D \to E$ is said to be decreasing (resp. increasing) if $x_1 \le x_2$ $(x_1, x_2 \in D)$ implies $Ax_2 \le Ax_1$ (resp. $Ax_1 \le Ax_2$). An operator $A: D \to E$ is said to be condensing if it is continuous, bounded, and $\gamma(A(S)) < \gamma(S)$, for any bounded set $S \subset D$ with $\gamma(S) > 0$, where $\gamma(S)$ denotes the measure of noncompactness of S.

For more details on these notions we refer to [1–3].

The following lemma is well known.

Lemma 1.1 ([1]). *Suppose that*

(i) *P is normal and operator $A: P \to P$ is decreasing and condensing;*

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(ii) *$A\theta > \theta$ and $A^2\theta > \epsilon_0 A\theta$, where $\epsilon_0 > 0$;*
(iii) *for any $x \geq \alpha A\theta$ ($\alpha = \alpha(x) > 0$) and $0 < t < 1$, there exists an $\eta = \eta(x, t) > 0$ such that*

$$A(tx) \leq [t(1+\eta)]^{-1}Ax.$$

Then A has exactly one positive fixed point $x^* > \theta$. Moreover, constructing successively sequence $x_n = Ax_{n-1}$ ($n = 1, 2, \ldots$) for any initial $x_0 \in P$, we have $\|x_n - x^*\| \to 0 \quad (n \to \infty)$.

In [4], Guo did not use the property that A is condensing. He replaced the above condition (iii) by the following condition

(iii′) for any $0 < a < b < 1$, there exists $\eta = \eta(a, b) > 0$ such that

$$A(tx) \leq [t(1+\eta)]^{-1}Ax, \quad \text{for all } a \leq t \leq b \text{ and } \theta < x \leq A\theta.$$

Guo obtained uniqueness of fixed points of A and convergence of the iterative sequence. Correlative researches on decreasing operators can be seen in [5–9] and the references therein.

Comparing the recent results with Lemma 1.1, compactness or continuity conditions of the decreasing operator A are weakened, but the condition (iii) is strengthened by similar condition (iii′) (that is, the operators satisfy some ordered inequality, such as some convexity, or concavity) such that the iterative sequences are convergent. In our results, conditions (i) and (ii) in Lemma 1.1 are weakened on different sides, condition (iii) is taken out outright. We obtain the existence of fixed points, uniqueness of fixed points and properties of the iterate sequences are studied too.

In order to discuss the main results, we need the following lemmas, which can be seen in [1–3].

Lemma 1.2. *Let $P \subset E$ be a normal cone, $u_0, v_0 \in E$, $u_0 < v_0$ and $B: [u_0, v_0] \to E$ be an increasing and condensing operator such that $u_0 \leq Bu_0$, $\quad Bv_0 \leq v_0$. Then B has a maximal fixed point u^* and a minimal fixed point v^* in $[u_0, v_0]$; moreover*

$$u^* = \lim_{n\to\infty} u_n, \qquad v^* = \lim_{n\to\infty} v_n, \tag{1.1}$$

where $u_n = Bu_{n-1}$, $v_n = Bv_{n-1}$ ($n = 1, 2, \ldots$), and

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

Lemma 1.3. *Let P be a normal cone in E. The following conclusions hold:*

(i) *$x_n \leq z_n \leq y_n$, ($n = 1, 2, \ldots$) and $\|x_n - x\| \to 0$, $\|y_n - x\| \to 0$ imply $\|z_n - x\| \to 0$;*
(ii) *every order interval $[x_1, x_2] = \{x | x_1 \leq x \leq x_2\}$ is bounded.*

Lemma 1.4. *Let $D \subset E$ be nonempty, closed bounded convex, and $A: D \to D$ be condensing. Then A has a fixed point.*

2. Main results and proof

Theorem 2.1. *Let E be a real Banach space, P a normal cone in E, $A: E \to E$ a decreasing and condensing operator. Assume that there exists $a \in E$ such that*

$$a \leq Aa, \qquad a \leq A^2 a. \tag{2.1}$$

Then A has a fixed point in $[a, Aa]$. Moreover

(i) *If $\{A^n a\}_{n=1}^{\infty}$ is convergent, then $w^* = \lim_{n\to\infty} A^n a$ is the unique fixed point of A in $S_-(Aa) \cup S_+(a)$;*
(ii) *If $\{A^n a\}_{n=1}^{\infty}$ is divergent, then there exist $u^*, v^* \in [A^2 a, Aa]$ such that*

$$u^* = \lim_{n\to\infty} A^{2n+2}a, \qquad v^* = \lim_{n\to\infty} A^{2n+1}a, \tag{2.2}$$

and for any fixed point x^* of A in $S_-(Aa) \cup S_+(a)$, the following inequality holds,

$$u^* < x^* < v^*. \tag{2.3}$$

Furthermore, if the cone P is minihedral, then x^* also satisfies

$$x^* \not< \min\left\{\frac{u^*+v^*}{2}, A\left(\frac{u^*+v^*}{2}\right)\right\}, \qquad x^* \not> \max\left\{\frac{u^*+v^*}{2}, A\left(\frac{u^*+v^*}{2}\right)\right\}. \tag{2.4}$$

Proof. The fact that $a \le Aa$ and A is decreasing implies

$$A^2a \le Aa, \qquad A^2a \le A^3a, \tag{2.5}$$

and A^2 is increasing. This, together with $A^2a \ge a$ gives

$$A^2a \le A^4a = A^2(A^2a), \qquad A^2(Aa) = A^3a \le Aa. \tag{2.6}$$

(2.5) and (2.6) imply

$$a \le A^2a \le A^4a \le \cdots \le A^{2n+2} \le \cdots \le A^{2n+1}a \le \cdots \le A^3a \le Aa. \tag{2.7}$$

Let

$$u_n = A^{2n+2}a, \qquad v_n = A^{2n+1}a,\ n = 0, 1, 2, \ldots.$$

By (2.7) we have

$$u_0 \le v_0, \qquad u_0 \le A^2u_0, \qquad A^2v_0 \le v_0, \tag{2.8}$$

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{2.9}$$

Since A^2 is increasing and condensing, Lemma 1.2, (2.8) and (2.9) imply A^2 (defined as B) has a maximal fixed point v^* and a minimal fixed point u^* in $[u_0, v_0]$, and (1.1) holds. Therefore we get

$$u^* = \lim_{n\to\infty} u_n = \lim_{n\to\infty} (A^2)^{n+1}a, \qquad v^* = \lim_{n\to\infty} v_n = \lim_{n\to\infty} (A^2)^n(Aa). \tag{2.10}$$

Let $\bar{x}$ be a fixed point of A^2. Then

$$A^2(A\bar{x}) = A^3\bar{x} = A(A^2\bar{x}) = A\bar{x}, \tag{2.11}$$

and thus $A\bar{x}$ is also a fixed point of A^2.

(i): $\{A^na\}_{n=1}^{\infty}$ is convergent. Then by (2.10) we know that $u^* = v^* = \lim\limits_{n\to\infty} A^na$, that is, fixed points of A^2 are unique in $[u_0, v_0]$, which we denote as w^*. By (2.11) we know that $w^* = Aw^*$, and thus A has a fixed point w^*. Since fixed points of A are fixed points of A^2, A has a unique fixed point $w^* = \lim_{n\to\infty} A^na$ in $[u_0, v_0]$.

Assume that $x^* \in S_-(Aa) \cup S_+(a)$ is a fixed point of A. If $x^* \ge a$, then

$$u_0 = A^2a \le A^2x^* = x^* = Ax^* \le Aa = v_0.$$

If $x^* \le Aa$, then $x^* = A^2x^* \ge A^2a = u_0$. Therefore, $x^* \in [u_0, v_0]$. This, together with the uniqueness of fixed points of A in $[u_0, v_0]$, implies that A has a unique fixed point $w^* = \lim_{n\to\infty} A^na$ in $S_-(Aa) \cup S_+(a)$.

(ii): $\{A^na\}_{n=1}^{\infty}$ is divergent. By (2.9) and (2.10) we know

$$u^* < v^*. \tag{2.12}$$

By (2.11), Au^* and Av^* are fixed points of A^2. The fact that A is decreasing implies

$$u^* \le Av^* \le Au^* \le v^*, \tag{2.13}$$

since u^*, v^* is the minimal fixed point and the maximal fixed point of A^2, respectively.

For any $x \in [u^*, v^*]$, we have $Av^* \le Ax \le Au^*$ as A is decreasing. This, together with (2.13), implies that $Ax \in [u^*, v^*]$. Therefore,

$$A[u^*, v^*] \subset [u^*, v^*]. \tag{2.14}$$

Since the order interval $[u^*, v^*]$ is a bounded closed convex set, (2.14) implies that A has a fixed point in $[u^*, v^*]$ by Lemma 1.4.

Let $x^* \in S_-(Aa) \cup S_+(a)$ be a fixed point of A. If $x^* \ge a$, then $x^* = Ax^* \le Aa = v_0$ for A is decreasing. If $x^* \le Aa$, then $x^* \ge A^2a \ge a$, that is, $x^* \in [u_0, v_0]$, which implies that all fixed points of A in $S_-(Aa) \cup S_+(a)$ are included in $[u_0, v_0]$. Since fixed points of A are fixed points of A^2 consequentially, every fixed point x^* of A in $S_-(Aa) \cup S_+(a)$ satisfies

$$u^* \le x^* \le v^*. \tag{2.15}$$

By (2.2) and continuity of A we know

$$v^* = \lim_{n\to\infty} A^{2n+1}a = A\left(\lim_{n\to\infty} A^{2n}a\right) = Au^*. \tag{2.16}$$

We can claim that u^* is not a fixed point of A. Otherwise, (2.16) implies $v^* = u^*$, which contradicts (2.12). Similarly, v^* is not a fixed point of A either. This, together with (2.15), implies that (2.3) holds.

Let

$$x_1 = \inf\left\{\frac{u^*+v^*}{2}, A\left(\frac{u^*+v^*}{2}\right)\right\}, \qquad x_2 = \sup\left\{\frac{u^*+v^*}{2}, A\left(\frac{u^*+v^*}{2}\right)\right\}$$

with u^*, v^* as (2.2). Therefore

$$x_1 \le \frac{u^*+v^*}{2},\ x_1 \le A\left(\frac{u^*+v^*}{2}\right), \qquad \frac{u^*+v^*}{2} \le x_2,\ A\left(\frac{u^*+v^*}{2}\right) \le x_2.$$

Thus,

$$Ax_1 \ge A\left(\frac{u^*+v^*}{2}\right) \ge x_1, \qquad Ax_2 \le A\left(\frac{u^*+v^*}{2}\right) \le x_2;$$

Let x^* be a fixed point of A. If $x^* < x_1$, then $x^* = Ax^* \ge Ax_1 \ge x_1$, which is a contradiction. If $x^* > x_2$, then $x^* = Ax^* \le Ax_2 \le x_2$, which is also a contradiction. Thus (2.4) holds. This completes the proof of Theorem 2.1. □

3. Corollaries and relative results

Corollary 3.1. *Let E be a real Banach space, $P \subset E$ a normal cone. Suppose that $A: E \to E$ is a decreasing condensing operator, and there exists $b \in E$ such that*

$$b \ge Ab, \qquad b \ge A^2 b. \tag{3.1}$$

Then A has a fixed point in $[Ab, b]$. Moreover

(i) *If $\{A^n b\}_{n=1}^{\infty}$ is convergent, then $w^* = \lim_{n\to\infty} A^n b$ is the unique fixed point of A in $S_-(b) \cup S_+(Ab)$;*
(ii) *If $\{A^n b\}_{n=1}^{\infty}$ is divergent, then there exist $u^*, v^* \in [A^3 b, A^2 b]$ such that*

$$u^* = \lim_{n\to\infty} A^{2n+1} b, \qquad v^* = \lim_{n\to\infty} A^{2n} b,$$

and

$$u^* < x^* < v^*$$

for any fixed point x^* of A in $S_-(b) \cup S_+(Ab)$.

Assume further that the cone P is minihedral, then (2.4) holds.

Proof. By condition (3.1) we know that $Ab \le A(Ab)$, $Ab \le A^3 b = A^2(Ab)$, therefore condition (2.1) in Theorem 2.1 is satisfied for $Ab = a$, thus A has a fixed point in $[Ab, A^2 b]$. We observe that $[Ab, A^2 b] \subset [Ab, b]$, thus all conclusions of the corollary hold. □

Theorem 3.1. *Let E be a real Banach space, $P \subset E$ a normal cone, $A: E \to E$ a decreasing condensing operator, and there exist $u_0, v_0 \in E$ such that*

$$u_0 \le Av_0 \le Au_0 \le v_0, \tag{3.2}$$

then A has a fixed point in $[u_0, v_0]$. Moreover

(i) *If $\{A^n u_0\}_{n=1}^{\infty}$ is convergent, and we write $w^* = \lim_{n\to\infty} A^n u_0$, then $\lim_{n\to\infty} A^n v_0 = w^*$, and w^* is the unique fixed point of A in $S_-(v_0) \cup S_+(u_0)$;*
(ii) *If $\{A^n u_0\}_{n=1}^{\infty}$ is divergent, then $\{A^n v_0\}_{n=1}^{\infty}$ is also divergent, there exist $u^*, v^* \in [u_0, v_0]$ such that*

$$u^* = \lim_{n\to\infty} A^{2n} u_0 = \lim_{n\to\infty} A^{2n+1} v_0, \qquad v^* = \lim_{n\to\infty} A^{2n-1} u_0 = \lim_{n\to\infty} A^{2n} v_0, \tag{3.3}$$

and

$$u^* < x^* < v^*, \tag{3.4}$$

for any fixed point x^* of A in $S_-(v_0) \cup S_+(u_0)$.

Assume further that the cone P is minihedral, then x^* still satisfies

$$x^* \not< \min\left\{\frac{u^*+v^*}{2}, A\left(\frac{u^*+v^*}{2}\right)\right\}, \qquad x^* \not> \max\left\{\frac{u^*+v^*}{2}, A\left(\frac{u^*+v^*}{2}\right)\right\}. \tag{3.5}$$

Proof. By condition (3.2) we obtain

$$u_0 \le Av_0 \le A^2u_0 \le A^2v_0 \le Au_0 \le v_0.$$

Hence, by induction, we have

$$\begin{aligned} u_0 &\le Av_0 \le A^2u_0 \le \cdots \le A^{2n}u_0 \le A^{2n+1}v_0 \le A^{2n+2}u_0 \\ &\le \cdots \le A^{2n+2}v_0 \le A^{2n+1}u_0 \le A^{2n}v_0 \le \cdots \le A^2v_0 \le Au_0 \le v_0, \quad n = 0, 1, 2 \ldots. \end{aligned} \tag{3.6}$$

Clearly, A^2 is an increasing condensing operator. Using (3.6) and Lemma 1.2 we know that A^2 has a maximal fixed point v^* and a minimal fixed point u^* in $[u_0, v_0]$. Moreover

$$u^* = \lim_{n\to\infty} A^{2n}u_0, \qquad v^* = \lim_{n\to\infty} A^{2n}v_0. \tag{3.7}$$

From (3.6) we can see that

$$A^{2n}u_0 \le A^{2n+1}v_0 \le A^{2n+2}u_0, \qquad A^{2n+2}v_0 \le A^{2n+1}u_0 \le A^{2n}v_0, \quad n = 0, 1, 2 \ldots. \tag{3.8}$$

Since the cone P is normal, from Lemma 1.3, (3.7) and (3.8) imply

$$\lim_{n\to\infty} A^{2n+1}v_0 = u^*, \qquad \lim_{n\to\infty} A^{2n+1}u_0 = v^*. \tag{3.9}$$

Next we consider two cases.

Case 1: $\{A^nu_0\}_{n=1}^{\infty}$ is convergent. From (3.7) and (3.9) we know that

$$u^* = v^* = \lim_{n\to\infty} A^nu_0 = \lim_{n\to\infty} A^nv_0,$$

that is, A^2 has a unique fixed point in $[u_0, v_0]$, which we denote as w^*. Clearly, $A^2(Aw^*) = A(A^2w^*) = Aw^*$, this is, Aw^* is also a fixed point of A^2, which implies A has a fixed point w^*, for fixed point of A^2 is unique. Since fixed points of A are also the fixed points of A^2, A has a unique fixed point w^* in $[u_0, v_0]$, where $w^* = \lim_{n\to\infty} A^nu_0 = \lim_{n\to\infty} A^nv_0$.

Assume that $x^* \in S_-(v_0) \cup S_+(u_0)$ is a fixed point of A. If $x^* \le v_0$, then (3.2) implies $x^* \ge Av_0 \ge u_0$; if $x^* \ge u_0$, then $x^* \le Au_0 \le v_0$. Therefore, $x^* \in [u_0, v_0]$. This combines with the uniqueness of fixed point of A in $[u_0, v_0]$ implies A has unique fixed point w^* in $S_-(b) \cup S_+(Ab)$, where $w^* = \lim_{n\to\infty} A^nu_0 = \lim_{n\to\infty} A^nv_0$.

Case 2: $\{A^nu_0\}_{n=1}^{\infty}$ is divergent. (3.7) and (3.9) imply that (3.3) holds and $u^* \ne v^*$. (3.6) and (3.7) imply

$$u^* < v^*. \tag{3.10}$$

By adopting similar method in Theorem 2.1, we can prove that A has a fixed point in $[u^*, v^*]$.

Assume that $x^* \in S_-(v_0) \cup S_+(u_0)$ is a fixed point of A. Similar to the proof in (i) we obtain $x^* \in [u_0, v_0]$. Clearly, x^* is also a fixed point of A^2, and thus

$$u^* \le x^* \le v^*. \tag{3.11}$$

By (3.7) and the continuity of A, we have

$$v^* = \lim_{n\to\infty} A^{2n+1}u_0 = A\left(\lim_{n\to\infty} A^{2n}u_0\right) = Au^*. \tag{3.12}$$

We can claim that u^* is not a fixed point of A. Otherwise, (3.12) implies $v^* = u^*$, which contradicts (3.10). Similarly, v^* is also not a fixed point of A. This and (3.11) imply that (3.4) holds. (3.5) can be proved in a similar way to Theorem 2.1. This completes the proof of Theorem 3.1. □

4. Applications in nonlinear second order elliptic equations

Let Ω be a bounded convex domain in R^n $(n \ge 2)$ whose boundary $\partial\Omega$ is assumed to be sufficiently smooth. Consider a uniformly elliptic differential operator on $\overline{\Omega}$:

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

i.e., there exists a positive constant μ_0 such that $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \mu_0|\xi|^2$, for any $x \in \overline{\Omega}$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in R^n$, and $a_{ij}(x) = a_{ji}(x)$, $c(x) \ge 0$. For the sake of simplicity, we shall assume that all functions $a_{ij}(x)$, $b_i(x)$, and $c(x)$ are sufficiently smooth.

Considering the Dirichlet problem

$$Lu = f(x, u), \quad u\big|_{\partial\Omega} = 0, \tag{4.1}$$

we have following conclusions.

Theorem 4.1. *Suppose that $f(x, u) \in C\left(\overline{\Omega} \times [0, +\infty), [0, +\infty)\right)$, which is decreasing on u, $f(x, \lambda) \not\equiv 0\ \forall \lambda \geq 0$. Then the problem (4.1) has a positive solution $w(x) \in C(\overline{\Omega})$ $(w(x) \geq 0,\ w(x) \not\equiv 0)$. Moreover, for initial function $u_0(x) \equiv 0$, we denote the unique solution of the linear Dirichlet problem*

$$Lu = f(x, u_{k-1}(x)), \quad u\big|_{\partial\Omega} = 0 \tag{4.2}$$

by $u_k(x)$, $k = 1, 2 \ldots$.

If $\{u_k(x)\}$ is convergent on $\overline{\Omega}$, then its convergence is uniform, and the limit function $w(x)$ is the unique solution of the boundary value problem (4.1).

If $\{u_k(x)\}$ is divergent on $\overline{\Omega}$, then there exist $u^*(x), v^*(x) \in C(\overline{\Omega})$ such that $u^*(x) \leq v^*(x)$, $x \in \overline{\Omega}$, and $\{u_{2k}(x)\}$ must converge to $u^*(x)$ uniformly on $\overline{\Omega}$, $\{u_{2k+1}(x)\}$ must converge to $v^*(x)$ uniformly on $\overline{\Omega}$. Any solution $w(x)$ of the Eq. (4.1) satisfies

$$u^*(x) \leq w(x) \leq v^*(x), x \in \overline{\Omega}, \quad w(x) \not\equiv u^*(x),\ w(x) \not\equiv v^*(x).$$

Proof. It is well known (see [1,10]) that the solution of the Dirichlet problem (4.1) is equivalent to the fixed point of the integral operator A:

$$Au(x) = \int_{\overline{\Omega}} G(x, y) f(y, u(y)) \mathrm{d}y, \tag{4.3}$$

where $G(x, y)$ denotes the Green function of differential operator L with boundary condition $u\big|_{\partial\Omega} = 0$. It is also well known that $G(x, y)$ satisfies the following inequality:

$$0 < G(x, y) < \begin{cases} K_0 |x - y|^{2-n}, & n > 2, \\ K_0 |\ln |x - y||, & n = 2, \end{cases} \quad (x, y \in \Omega,\ x \neq y). \tag{4.4}$$

Hence, the linear integral operator

$$Bv(x) = \int_{\overline{\Omega}} G(x, y) v(y) \mathrm{d}y$$

is a completely continuous operator from $C(\overline{\Omega})$ into $C(\overline{\Omega})$. Clearly, superposition operator $F\phi(x) = f(x, \phi(x))$ that maps P into P is continuous and bounded. Where $P = \{u(x) \in C(\overline{\Omega}) \big| u(x) \geq 0, \forall x \in C(\overline{\Omega})\}$ is a normal minihedral cone of space $C(\overline{\Omega})$. Therefore, operator $A = BF$ that maps P into P is completely continuous, and thus A is condensing.

Evidently, from $f(x, 0) \not\equiv 0$ and (4.4) we have

$$A\theta = \int_{\overline{\Omega}} G(x, y) f(y, 0) \mathrm{d}s > \theta; \tag{4.5}$$

Since $A\theta = \int_{\overline{\Omega}} G(x, y) f(y, 0) \mathrm{d}s \in C(\overline{\Omega})$, there exists a positive constant C such that

$$A\theta = \int_{\overline{\Omega}} G(x, y) f(y, 0) \mathrm{d}s \leq C. \tag{4.6}$$

This, together with (4.3) and (4.6), implies

$$A^2\theta = A(A\theta) \geq \int_{\overline{\Omega}} G(x, s) f(s, C) \mathrm{d}s > \theta, \tag{4.7}$$

for A decreasing. (4.5) and (4.7) imply that A satisfies all conditions for $a = \theta$ in Theorem 2.1. Thus, A has a positive fixed point $w, \theta < w < A\theta$. That is, there exists a function $w(t) \in C(\overline{\Omega})$, $w(x) \geq 0$, $w(x) \not\equiv 0$, which satisfies (4.1).

The sequence $\{A^n a\}_{n=0}^{\infty}$ in Theorem 2.1 is equivalent to

$$u_0(x) \equiv 0, \qquad u_k(x) = \int_{\overline{\Omega}} G(x, y) f(y, u_{k-1}(y)) \mathrm{d}y, \quad k = 1, 2, \ldots. \tag{4.8}$$

(4.8) is equivalent to

$$u_0(x) \equiv 0, \qquad Lu_k(x) = f(x, u_{k-1}(x)), \quad u_k(x)\big|_{\partial\Omega} = 0, \quad k = 1, 2, \ldots.$$

i.e., $u_k(x)$ is the unique solution of the linear Dirichlet problem (4.2).

Using Theorem 2.1 we know that conclusions of Theorem 4.1 hold. □

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